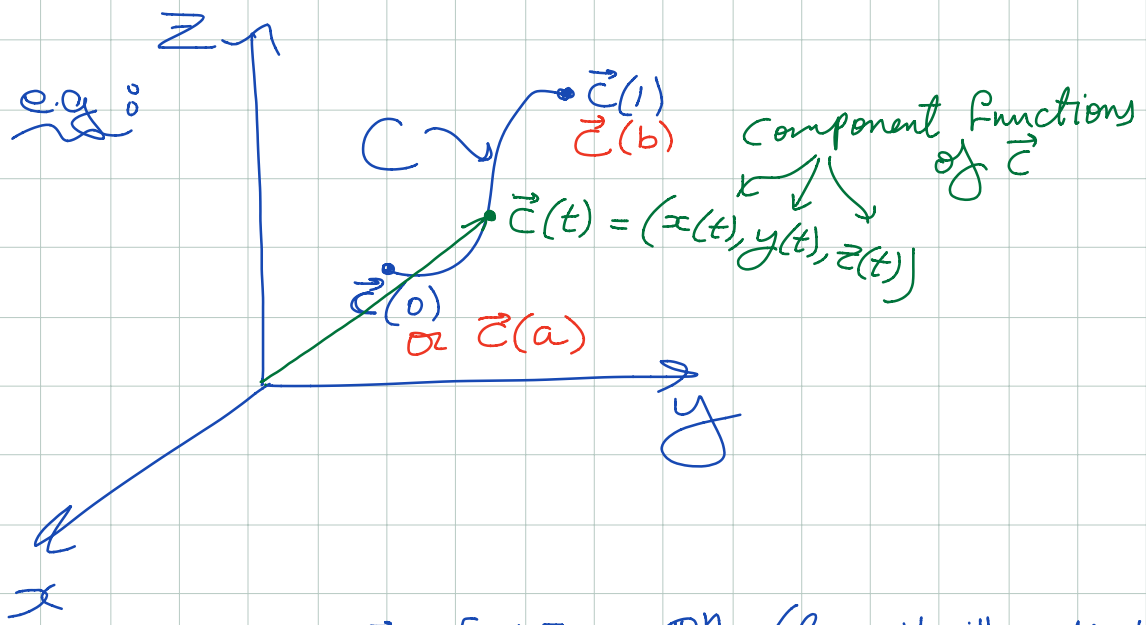
Section 2.4 Paths & Curves



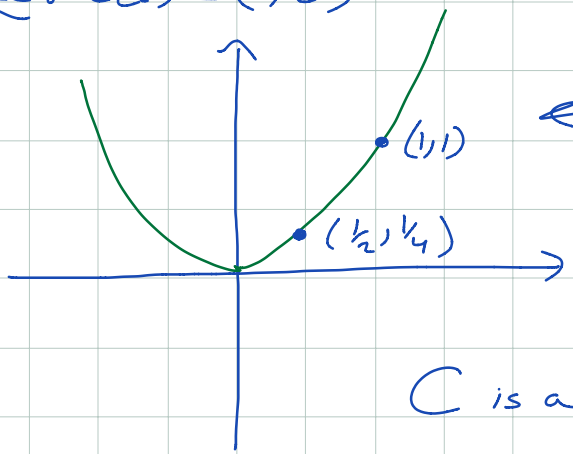
So $\vec{c}: [a,b] \longrightarrow \mathbb{R}^n$ (for us it will mostly be $\mathbb{R}^2$ & $\mathbb{R}^3$)

is called a path (in the plane when $n=2$, in space when $n=3$)

C is the collection of points $\vec{c}(t)$ as t varies in $[a,b]$

C is called a curve. $\vec{c}(a)$ & $\vec{c}(b)$ are its endpoints.

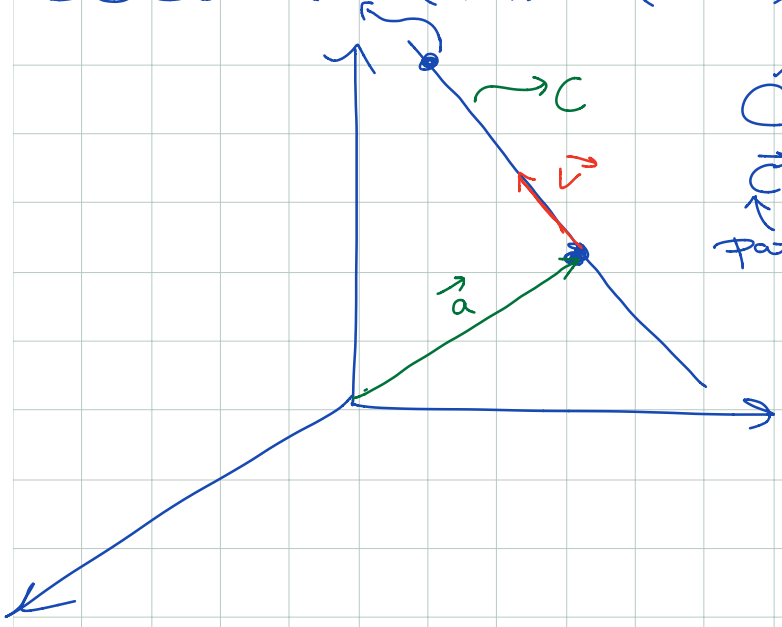
Example: $\vec{c}(t) = (t, t^2)$ so here $x(t) = t$, $y(t) = t^2$



← i.e. $y = x^2$

C is a parabolic arc

Example: $\vec{c}(t) = (a_1, a_2, a_3) + t(v_1, v_2, v_3)$



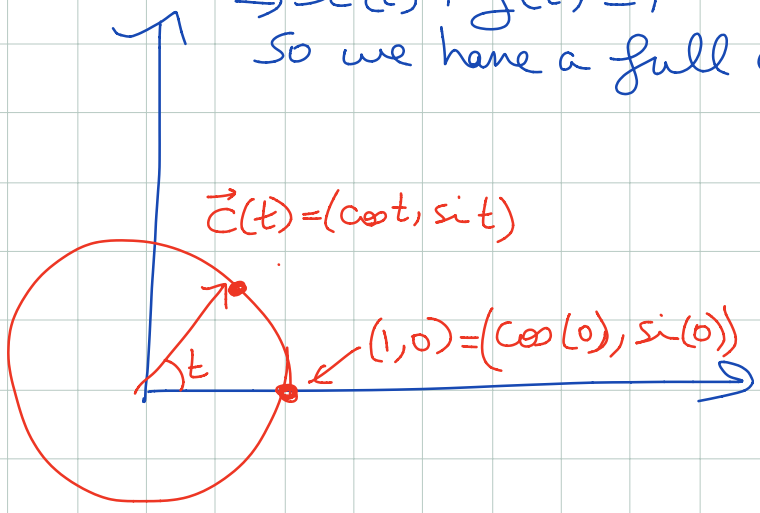
curve
C is a line
$\vec{c}$ traces out a line
path

Example: $\vec{c}(t) = (\cos t, \sin t)$ for $t \in [0, 2\pi]$

so $x(t) = \cos(t)$, $y(t) = \sin t$

$\Rightarrow x^2(t) + y^2(t) = 1$

So we have a full circle (t goes from 0 to 2π)



We usually think of t as time so $\vec{c}(t)$ is position at time t. Then we can talk about $\vec{v}(t)$ the velocity at time t.

$\vec{c}'(t)$

If $\vec{c}$ is a path & it is differentiable (remember what this means) we say $\vec{c}$ is a differentiable path.

The velocity of $\vec{c}$ at time t is given by

$$\vec{c}'(t) = \lim_{h \to 0} \frac{\vec{c}(t+h) - \vec{c}(t)}{h}$$

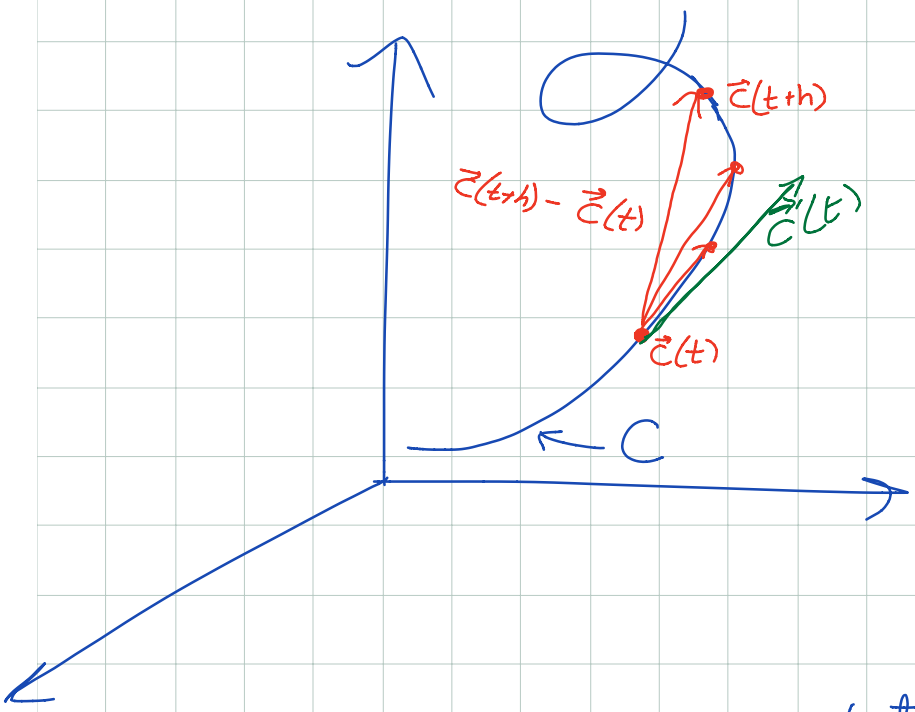
The speed is $\|\vec{c}'(t)\|$

If $\vec{c}(t) = (x(t), y(t), z(t))$

then $\vec{c}'(t) = (x'(t), y'(t), z'(t))$

$$= x'(t)\vec{i} + y'(t)\vec{j} + z'(t)\vec{k}$$

Note that: $\vec{c}'(t)$ is tangent to the path $\vec{c}(t)$ at time t.



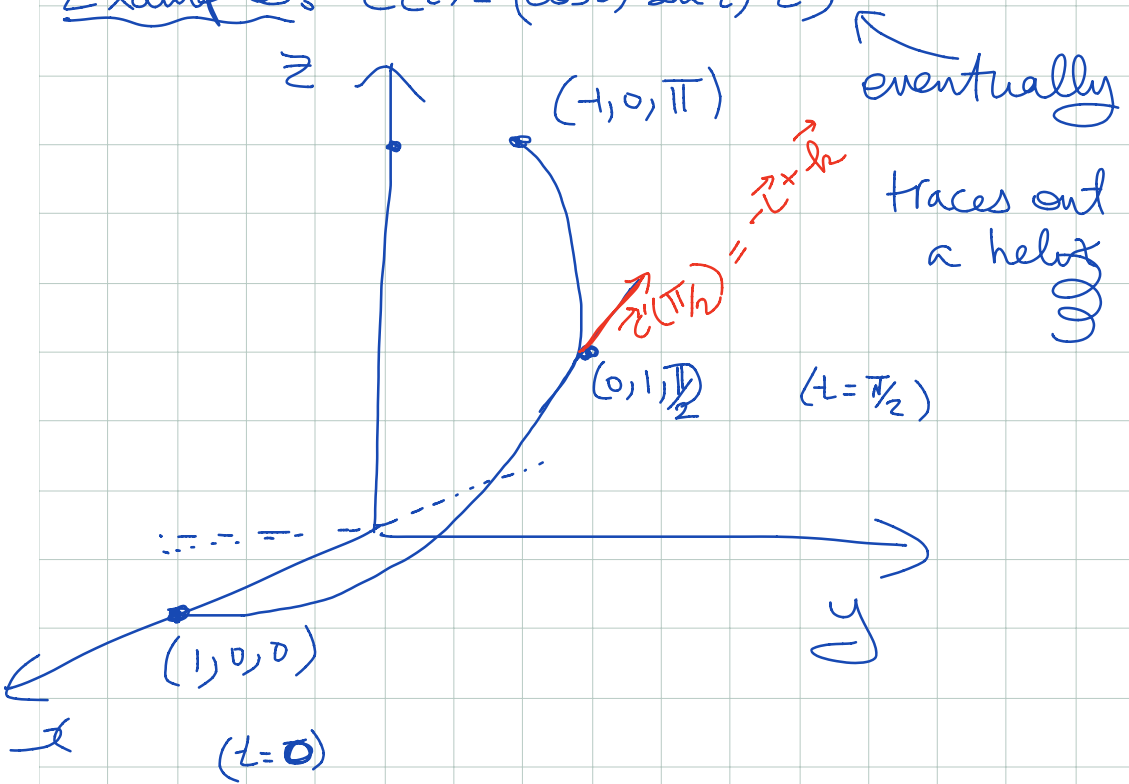
Example: Compute the tangent vector to the path

$\vec{c}(t) = (t, t+t^2, e^{2t})$ at $t = 1$

Sol'n: $\vec{c}'(t) = (1, 1+2t, 2e^{2t})$

$\Rightarrow \vec{c}'(1) = (1, 3, 2e^2)$

Example: $\vec{r}(t) = (\cos t, \sin t, t)$



$\vec{r}'(t) = (-\sin t, \cos t, 1)$

$\Rightarrow \vec{r}'(\pi/2) = (-1, 0, 1) = -\vec{i} + \vec{k}$

Tangent line to a path

Suppose $\vec{c}(t)$ is a path & $\vec{c}'(t_0) \neq 0$ at t_0 the equation of the tangent line at $\vec{c}(t_0)$

is $\vec{\ell}(t) = \vec{c}(t_0) + (t - t_0)\vec{c}'(t_0)$

Example: Suppose that a particle follows the path $\vec{c}(t) = (\cos t, \sin t, t)$ and flies off at a tangent at $t = \frac{\pi}{2}$. Where is it at $t = \pi/2 + 1$

Solution: We need to find the equation of the line tangent to the curve at $\vec{c}(\frac{\pi}{2}) = (0, 1, \frac{\pi}{2})$

Ok, $\vec{c}'(t) = (-\sin t, \cos t, 1)$ (t_0)

So $\vec{c}'(\pi/2) = (-1, 0, 1)$

$\Rightarrow \ell(t) = (0, 1, \pi/2) + (t - \pi/2)(-1, 0, 1)$

$\Rightarrow$ at $t = \pi/2 + 1$, $\ell(\pi/2 + 1) = (0, 1, \pi/2) + (\frac{\pi}{2} + 1 - \frac{\pi}{2})(-1, 0, 1)$

$\ell(\pi/2 + 1) = (0, 1, \pi/2) + (-1, 0, 1) = (-1, 1, \pi/2 + 1)$

2.5 Properties of the Derivative.

Goal: To differentiate products, quotients, sums of functions & compositions of functions

⇓

Chain rule

Let's start with sums, products & quotients

Recall that if $f: \mathbb{R}^n \to \mathbb{R}^m$ its derivative (total derivative) is $T = Df$, the matrix of partial derivatives.

Let us now assume the functions we have are differentiable

Rules: • If $h(\vec{x}) = c f(\vec{x})$

then $(Dh)(\vec{x}) = c(Df)(\vec{x})$

(constant multiple rule)

e.g. Let $f(x,y,z) = 3x^2 + 2yz$

and $h(x,y,z) = 6x^2 + 4yz = 2f(x,y,z)$

$$(Df)(x,y,z) = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right] = [6x, 2z, 2y]$$

$$(Dh)(x,y,z) = 2(Df)(x,y,z) = [12x, 4z, 4y]$$

- If $h(\vec{x}) = f(\vec{x}) + g(\vec{x})$

Then $(Dh)(\vec{x}) = (Df)(\vec{x}) + (Dg)(\vec{x})$

(Sum rule)

e.g.

$$f(x,y,z) = 3x^2 + 2yz$$

$$g(x,y,z) = e^{xyz}$$

$$h(x,y,z) = f(x,y,z) + g(x,y,z)$$

$$= 3x^2 + 2yz + e^{xyz}$$

$$(Dh)(x,y,z) = \underbrace{(6x, 2z, 2y)}_{Df} + \underbrace{(yze^{xyz}, xze^{xyz}, xye^{xyz})}_{Dg}$$

• Let $f: \mathbb{R}^n \to \mathbb{R}$, $g: \mathbb{R}^n \to \mathbb{R}$

Let $h(\vec{x}) = f(\vec{x})\, g(\vec{x})$

then $(Dh)(\vec{x}) = g(\vec{x})(Df)(\vec{x}) + f(\vec{x})(Dg)(\vec{x})$

Product rule

e.g.: $f(x,y) = x^2 + y^2$

$g(x,y) = xy$

Let $h(x,y) = f(x,y)\, g(x,y)$

$=$

$(Dh)(x,y) = g(x,y)(Df)(x,y) + f(x,y)(Dg)(x,y)$

$= (xy)(2x, 2y) + (x^2+y^2)(y, x)$

$= (2x^2y + xy^2 + y^3,\ 2xy^2 + x^3 + xy^2)$

$= (3x^2y + y^3,\ 3xy^2 + x^3)$

Check: $h(x,y) = x^3y + xy^3 \Rightarrow (Dh)(x,y) = (3x^2y + y^3,\ 3xy^2 + x^3)$

- Now let $h(\vec{x}) = f(\vec{x})/g(\vec{x})$ (where $g(x) \neq 0$)

then $(Dh)(\vec{x}) = \dfrac{g(\vec{x})(Df)(\vec{x}) - f(\vec{x})(Dg)(\vec{x})}{(g(\vec{x}))^2}$

(quotient rule)

Now, let's do the chain rule

Suppose $f: \mathbb{R}^n \to \mathbb{R}^m$ & $g: \mathbb{R}^m \to \mathbb{R}^p$ both differentiable

If we define $h(\vec{x}) = \underbrace{g(\overbrace{f(\vec{x})}^{\mathbb{R}^m})}_{\in \mathbb{R}^p}$ ($\vec{x} \in \mathbb{R}^n$)

i.e., h is the composition of f & g

i.e. $h = g \circ f$

what is $(Dh)(\vec{x})$?

Ans: $(Dh)(\vec{x}) = (D\vec{g})(F(\vec{x}))(DF)(x)$

Derivative of F evaluated at x

Derivative of g evaluated at $F(\vec{x})$

Possibly nicer notation: when

$$(Dh)\big|_{\vec{x}} = (Dg)\big|_{F(\vec{x})} \; (DF)\big|_{\vec{x}}$$

matrix

matrix

Chain rule for differentiating $h = g \circ F$

This should remind you of the single variable chain rule

$$\frac{d\, g(F(x))}{dx} = \left(g'(F(x))\right) F'(x)$$

In this course, we will be interested in two special cases of the chain rule

Special Case (I)

$\vec{c}: \mathbb{R} \to \mathbb{R}^3$ is a differentiable path

$f: \mathbb{R}^3 \to \mathbb{R}$

$h(t) = f(\vec{c}(t)) = f(x(t), y(t), z(t))$

want $\frac{dh}{dt}$.

We'll need the following definition

Definition: If $f: \mathbb{R}^n \to \mathbb{R}$, $(Df)(\vec{x})$ is a $1 \times n$ matrix $\left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots \frac{\partial f}{\partial x_n}\right]$

We can form the corresponding vector

$\left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}\right)$, called the gradient of f & denoted ∇f. Note that ∇f is a vector

$$\frac{dh}{dt} = (Df)(\vec{c}(t)) \cdot \vec{c}'(t)$$

$$= \underbrace{\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)\Big|_{(x(t), y(t), z(t))}}_{\nabla f|_{(x(t), y(t), z(t))}} \cdot \overbrace{\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)}^{\vec{c}'(t)}$$

$$\frac{dh}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

$$\Updownarrow$$

$$\frac{dh}{dt} = \nabla f|_{\vec{c}(t)} \cdot \vec{c}'(t)$$

→ vector form

dot product

Example: Suppose that a particle moves along the path

$\vec{c}(t) = (t, 2t, 3t)$ and that the temperature of a point (x, y, z) is given by $f(x, y, z) = \cos x + \sin y + \cos z$

(1) What is the temp. experienced by the particle as a function of time?

Solution: The temp. experienced by the particle is given by

$$g(t) = f(\vec{c}(t)) = f(t, 2t, 3t) = \cos t + \sin 2t + \cos 3t$$

(2) Use the chain rule to determine the rate of change of temp. as experienced by the particle.

Solution: we want $g'(t) = \frac{dg}{dt}$

$$= \nabla f|_{\vec{c}(t)} \cdot \vec{c}'(t)$$

but

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

$$= (-\sin x, \cos y, -\cos z)$$

so $\nabla f|_{\vec{c}(t)} = (-\sin t, \cos(2t), -\sin(3t))$

$$\nabla f|_{\vec{c}(t)} \cdot \vec{c}'(t) = (-\sin t, \cos t, -\sin 3t) \cdot (1, 2, 3)$$

$$= -\sin t + 2\cos t - 3\sin 3t$$

Case (II)

$$f: \mathbb{R}^3 \to \mathbb{R} \quad \& \quad g: \mathbb{R}^3 \to \mathbb{R}^3$$

we write

$$g(x,y,z) = (u(x,y,z), v(x,y,z), w(x,y,z))$$

and define $h: \mathbb{R}^3 \to \mathbb{R}$ by $h = f \circ g$ i.e.,

$$h(x,y,z) = f(u(x,y,z), v(x,y,z), w(x,y,z))$$

$$\Rightarrow Dh = Df|_{(u,v,w)} \, Dg$$

$$\left[\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}, \frac{\partial h}{\partial z}\right] = \left[\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}, \frac{\partial f}{\partial w}\right] \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix}$$

What this really means:

$$\frac{\partial h}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x}$$

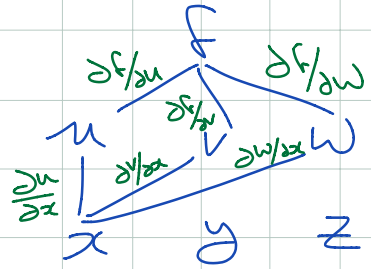
$$\frac{\partial h}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial y}$$

$$\frac{\partial h}{\partial z} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial z} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial z}$$

"Easy" way to remember

$$h(x,y,z) = f(u(x,y,z), v(x,y,z), w(x,y,z))$$

to get $\frac{\partial h}{\partial x}$:



Informally, there are 3 routes to get from f to x

$$\frac{\partial h}{\partial x} = \underbrace{\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}}_{\text{route 1}} + \underbrace{\frac{\partial f}{\partial v}\frac{\partial v}{\partial x}}_{\text{route 2}} + \underbrace{\frac{\partial f}{\partial w}\frac{\partial w}{\partial x}}_{\text{route 3}}$$

Example: Verify the chain rule for computing $\frac{\partial h}{\partial x}$ where

$$h(x,y,z) = f(u(x,y,z), v(x,y,z), w(x,y,z))$$

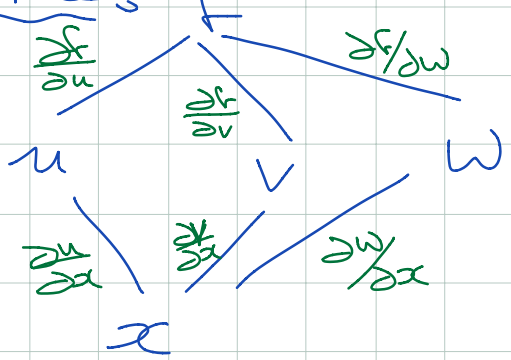
where $f(u,v,w) = u^2 + v^2 - w$

and $u(x,y,z) = x^2 y$

$v(x,y,z) = yz$

$w(x,y,z) = e^{-xz}$

Soln Chain rule:



$$\frac{\partial h}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x}$$

$$= (2u)(2xy) + (2v)(0) + (-1)(-ze^{-xz})$$

$$= (2x^2y)(2xy) + ze^{-xz}$$

Directly differentiating

$$h(x,y,z) = (x^2y)^2 + y^4 - e^{-xz}$$

$$\Rightarrow \frac{\partial h}{\partial x} = 4x^3y^2 + ze^{-xz} \Rightarrow$$

Another example:

Polar Coordinates:

Let $f(x,y)$ be some function

and make the substitution $x = r\cos\theta$

$y = r\sin\theta$

Find $\frac{\partial f}{\partial \theta}$

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta}$$

$$= -r\sin\theta \frac{\partial f}{\partial x} + r\cos\theta \frac{\partial f}{\partial y}$$

Section 2.6 : Gradients & Directional Derivatives

Recall: If $f: \mathbb{R}^3 \to \mathbb{R}$ is differentiable its gradient is the vector given by

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

e.g.

$$f(x,y,z) = xyz + e^x$$

$$\Rightarrow \nabla f = (yz + e^x, xz, xy)$$

Directional Derivatives

Recall that if we have $f(x,y,z)$

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h,y,z) - f(x,y,z)}{h}$$

$$= \lim_{h \to 0} \frac{f(\vec{x} + h\vec{i}) - f(\vec{x})}{h}$$

So $\frac{\partial f}{\partial x}$ is the derivative in the direction of $\vec{i}$

Similarly $\frac{\partial f}{\partial y}$ is the derivative in the direction of $\vec{j}$.

What if we want the derivative in the direction of some unit vector $\vec{v}$?

$$\lim_{h\to 0} \frac{f(\vec{x}+h\vec{v}) - f(\vec{x})}{h}$$

Directional Derivative in the direction of the unit vector $\vec{v}$.

in the dir. of $\vec{v}$, $\|\vec{v}\|=1$

Theorem: The Directional Derivative of a differentiable function $f: \mathbb{R}^3 \to \mathbb{R}$ is

Dir. Der.: $(Df)(\vec{x})\vec{v} = \underbrace{\nabla f}_{\text{vector}} \cdot \underbrace{\vec{v}}_{\text{vector}}$ ← dot product

$$= \frac{\partial f}{\partial x}(\vec{x})v_1 + \frac{\partial f}{\partial y}(\vec{x})v_2 + \frac{\partial f}{\partial z}(\vec{x})v_3$$

Proof: The dir. derivative is

$$\lim_{t \to 0} \frac{f(\vec{x}+t\vec{v}) - f(\vec{x})}{h}$$

defining $\vec{c}(t) = \vec{x} + t\vec{v}$

the directional der. can be written as $\lim_{t \to 0} \frac{f(\vec{c}(t)) - f(\vec{c}(0))}{t} = g'(0)$

where $g(t) = f(\vec{c}(t))$

but by the chain rule

$$g'(t) = (\nabla f)\Big|_{\vec{c}(t)} \cdot \vec{c}'(t)$$

$$\Rightarrow g'(0) = (\nabla f)\Big|_{\vec{c}(0)} \cdot \vec{c}'(0)$$

$\vec{c}'(0) \to \vec{v}$

$\vec{c}(0) = x$

$$= (\nabla f)(x) \cdot \vec{v}$$

Example: $f(x,y,z) = x^2 e^{-yz}$

Find the rate of change of f in the direction of $\vec{v} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ at the point $(1,0,0)$

Solution: We need

$$\nabla f \cdot \vec{v} = \left(2xe^{-yz}, -x^2 z e^{-yz}, -x^2 y e^{-yz}\right) \cdot \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$\Rightarrow (\nabla f \cdot \vec{v})\big|_{(1,0,0)} = (2,0,0) \cdot \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = \frac{2}{\sqrt{3}}.$$

Remark: If $\vec{v}$ is not a unit vector, use $\frac{\vec{v}}{\|\vec{v}\|}$.

Directions of Fastest Increase

If $\vec{n}$ is a unit vector, the rate of change of f in the direction of $\vec{n}$ is

$$(\nabla f)(\vec{x}) \cdot \vec{n} = \|(\nabla f)(\vec{x})\| \underbrace{\|\vec{n}\|}_{=1} \underbrace{\cos\theta}_{|\cos\theta| \le 1}$$

So if $\nabla f(\vec{x}) \neq 0$, the rate of change is maximized when $\cos\theta = 1$, i.e., $\theta = 0$ $\Leftrightarrow$ when $\vec{n}$ & ∇f are parallel.

In summary, If $\nabla f(x) \neq 0$, then $\nabla f(x)$ points in the direction along which f is increasing the fastest

Example: Find the direction of fastest increase (maximum rate of change) of $f(x,y) = x^2 + y^3$ at the point $P = (2,3)$

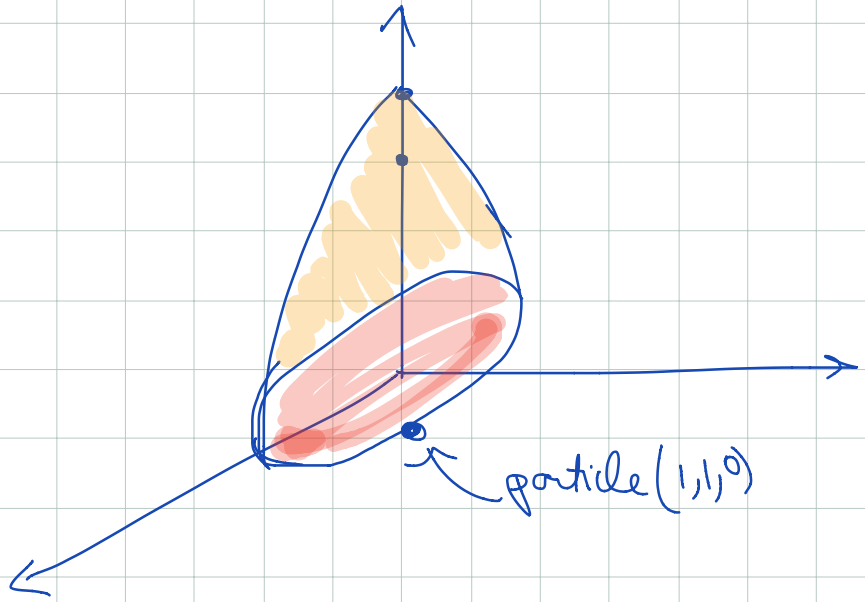
Soln: $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (2x, 3y^2)$

$\Rightarrow (\nabla f)(2,3) = (4, 27)$

So the direction of maximum increase is $4\vec{i} + 27\vec{j}$

Example:

A particle is at the point $(1,1,0)$ on the surface given by $f(x,y) = 4 - x^2 - 3y^2$



- Find ∇f.

ans: $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (-2x, -6y)$

- What is the direction of maximal increase for the particle

ans. $(\nabla f)(1,1) = (-2, -6)$

(So if the particle wanted to climb the mountain it should do so by moving in this direction.)

• What is the directional derivative of f in the direction $(1,1)$?

$$(\nabla f)(\vec{x}) \cdot \frac{\vec{u}}{\|\vec{u}\|} = (2x, -6y) \cdot \frac{(1,1)}{\sqrt{2}}$$

$$= \frac{2x - 6y}{\sqrt{2}}$$

and in the direction of the gradient?

$$(\nabla f)(\vec{x}) \cdot \frac{(\nabla f)(\vec{x})}{\|(\nabla f)(\vec{x})\|} = \|(\nabla f)(\vec{x})\| = \sqrt{4x^2 + 36y^2}$$

Chapter 3: Higher order derivatives; Maxima & Minima

Recall: In single variable calculus we used the derivative $f'(x)$ to test for critical points ($f'(x_0) = 0$) and we checked $f''(x)$ to see if x_0 is a max ($f''(x_0) < 0$) or a min ($f''(x_0) < 0$).

Goal: Extend the methods to real valued functions of several variables ($f: \mathbb{R}^m \to \mathbb{R}$), so we have to develop higher order derivatives and derive tests for maxima, minima, & saddle points

3.1 Iterated partial derivatives

Let $f: \mathbb{R}^3 \to \mathbb{R}$ have continuous partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ & $\frac{\partial f}{\partial z}$. We call such a function a C^1 function.

If each of these partials themselves have continuous partials, we say that f is a C^2 function.

Notation:

Iterated Partial derivatives

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

...

mixed Partial derivatives

Example:

Find all the second partial derivatives of

$$f(x,y) = x^2y^3 + e^x$$

Soln

$$\frac{\partial f}{\partial x} = 2xy^3 + e^x \qquad \frac{\partial f}{\partial y} = 3x^2y^2$$

$$\Rightarrow \frac{\partial^2 f}{\partial x^2} = 2y^3 + e^x \;, \qquad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}(3x^2y^2) = 6x^2y$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}(3x^2y^2) = 6xy^2$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}(2xy^3 + e^x) = 6xy^2$$

Example: $f(x,y) = \cos x \sin y$

Sol'n: $\frac{\partial f}{\partial x} = -\sin x \sin y \;, \quad \frac{\partial f}{\partial y} = \cos x \cos y$

$$\frac{\partial^2 f}{\partial x^2} = -\cos x \sin y \;, \quad \frac{\partial^2 f}{\partial y^2} = -\cos x \sin y$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}(+\cos x \sin y) = -\sin x \cos y$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}(-\sin x \sin y) = -\sin x \cos y$$

Remark: In both examples, we had

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

Theorem: If $f(x,y)$ is of class C^2 then the mixed partials are equal, i.e.,

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

equality of mixed partials

Exercise: prove it / read the proof in book

Example: Let $w = f(x,y)$ where $x = u+v$, $y = u-v$

Find $\dfrac{\partial^2 w}{\partial u \partial v}$

Solution: $\dfrac{\partial^2 w}{\partial u \partial v} = \dfrac{\partial}{\partial u}\left(\dfrac{\partial w}{\partial v}\right)$

Chain rule →
$$= \frac{\partial}{\partial u}\left(\frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v}\right)$$

$$= \frac{\partial}{\partial u}\left(\frac{\partial w}{\partial x}\cdot 1 + \frac{\partial w}{\partial y}(-1)\right)$$

$$= \frac{\partial}{\partial u}\underbrace{\frac{\partial w}{\partial x}}_{g(x,y)} - \frac{\partial}{\partial u}\underbrace{\frac{\partial w}{\partial y}}_{h(x,y)}$$

another two chain rules →
$$= \frac{\partial g}{\partial x}\underbrace{\frac{\partial x}{\partial u}}_{1} + \frac{\partial g}{\partial y}\underbrace{\frac{\partial y}{\partial u}}_{1} - \frac{\partial h}{\partial x}\underbrace{\frac{\partial x}{\partial u}}_{1} - \frac{\partial h}{\partial y}\underbrace{\frac{\partial y}{\partial u}}_{1}$$

$$= \frac{\partial g}{\partial x} + \frac{\partial g}{\partial y} - \frac{\partial h}{\partial x} - \frac{\partial h}{\partial y}$$

substitute $\frac{\partial w}{\partial x} = g$, $\frac{\partial w}{\partial y} = h$ →
$$= \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial x \partial y} - \frac{\partial^2 w}{\partial y \partial x} - \frac{\partial^2 w}{\partial y^2}$$

$$= \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2}$$ ∎